ℓ_∞-SUMS AND THE BANACH SPACE ℓ_∞/c_0

CHRISTINA BRECH AND PIOTR KOSZMIDER

Abstract. This paper is concerned with the isomorphic structure of the Banach space ℓ_∞/c_0 and how it depends on combinatorial tools whose existence is consistent but not provable from the usual axioms of ZFC. Our main global result is that it is consistent that ℓ_∞/c_0 does not have an orthogonal ℓ_∞-decomposition that is, it is not of the form $\ell_\infty(X)$ for any Banach space X. The main local result is that it is consistent that $\ell_\infty(c_0(\mathfrak{c}))$ does not embed isomorphically into ℓ_∞/c_0, where $\mathfrak{c}$ is the cardinality of the continuum, while ℓ_∞ and $c_0(\mathfrak{c})$ always do embed quite canonically. This should be compared with the results of Drewnowski and Roberts that under the assumption of the continuum hypothesis ℓ_∞/c_0 is isomorphic to its ℓ_∞-sum and in particular it contains an isomorphic copy of all Banach spaces of the form $\ell_\infty(X)$ for any subspace X of ℓ_∞/c_0.

1. introduction

Drewnowski and Roberts proved in [5] that, assuming the Continuum Hypothesis (abbreviated CH), the Banach space ℓ_∞/c_0 is isomorphic to its ℓ_∞-sum denoted $\ell_\infty(\ell_\infty/c_0)$. They concluded that under the assumption of CH the Banach space ℓ_∞/c_0 is primary, that is, given a decompositon $\ell_\infty/c_0 = A \oplus B$, one of the spaces A or B must be isomorphic to ℓ_∞/c_0. The proof relies on the Pełczyński decomposition method and on another striking result from [5] (not requiring CH) which says that one of the factors A or B as above must contain a complemented subspace isomorphic to ℓ_∞/c_0. Another conclusion was that $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$ is isomorphic to ℓ_∞/c_0 under the assumption of CH.

In this paper we show that some of the above statements cannot be proved without some additional set theoretic assumptions. Namely, for any cardinal $\kappa \geq \omega_2$, the following statements all hold in the Cohen model obtained by adding κ-many Cohen reals to a model of CH ($\mathfrak{c}$ denotes the cardinality of the continuum):

(a) $\ell_\infty(c_0(\omega_2))$ does not embed isomorphically into ℓ_∞/c_0,
(b) $\ell_\infty(c_0(\mathfrak{c}))$ does not embed isomorphically into ℓ_∞/c_0,
(c) $\ell_\infty(\ell_\infty/c_0)$ does not embed isomorphically into ℓ_∞/c_0,
(d) ℓ_∞/c_0 is not isomorphic to $\ell_\infty(X)$ for any Banach space X,
(e) $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$ is not isomorphic to ℓ_∞/c_0.

Below we show that (a) easily implies the other statements and so later we will focus on proving (a). Indeed, (a) implies (b) simply because $\mathfrak{c} \geq \omega_2$ in those models. (c) follows from (b) and the fact that ℓ_∞/c_0 contains an isometric copy of $c_0(\mathfrak{c})$ (e.g., the closure of the space spanned by the classes of characteristic functions of elements

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of a family $\{A_\xi : \xi < \mathfrak{c}\}$ of infinite subsets of $\mathbb{N}$ whose pairwise intersections are finite). To conclude (d) from (a), use a result of Rosenthal (see Theorem 7.11 of [9]) that if $T : c_0(\Gamma) \to X$ is a bounded linear operator such that $|\{\gamma \in \Gamma : |T(\chi_{\{\gamma\}})| > \varepsilon\}| = |\Gamma|$ for some $\varepsilon > 0$, then there is $\Gamma' \subseteq \Gamma$ such that $|\Gamma'| = |\Gamma|$ and T restricted to $c_0(\Gamma')$ is an isomorphism onto its image; hence if $\ell_\infty(X)$ contains $c_0(\omega_2)$, then so does X. Finally (e) follows from (c) alone, because $\ell_\infty(\ell_\infty/c_0)$ embeds isometrically into $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$. Indeed, consider a partition of $\mathbb{N}$ into pairwise disjoint infinite sets $(A_i : i \in \mathbb{N})$ and for each $x \in \ell_\infty(\ell_\infty/c_0)$ consider $x' \in \ell_\infty(\ell_\infty/c_0)$ such that $x'(n) = x(i)$ if and only if $n \in A_i$. Note that $T : \ell_\infty(\ell_\infty/c_0) \to \ell_\infty(\ell_\infty/c_0)$ given by $T(x) = x'$ is an isometric embedding. Moreover it gives an isometric embedding while composed with the quotient map from $\ell_\infty(\ell_\infty/c_0)$ onto $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$.

We emphasize an interesting phenomenon that follows from the gap which may exist between the number of added Cohen reals and ω_2: even when $\mathfrak{c}$ is very large, meaning that ℓ_∞/c_0 has large density, still it may not contain an isomorphic copy of $\ell_\infty(c_0(\omega_2))$ while it always contain quite canonical copies of both ℓ_∞ and $c_0(\omega_2)$.

It remains unknown if ℓ_∞/c_0 is primary in the above models and in general if the primariness of ℓ_∞/c_0 can be proved without additional set theoretic assumptions. It would also be interesing to conclude the above statements in a more axiomatic way as in [16] or [12].

Another problem mentioned in [5] remains open as well (including in the Cohen model), namely if ℓ_∞/c_0 has the Schroeder-Bernstein property, that is if there exists a complemented subspace X of ℓ_∞/c_0, nonisomorphic to ℓ_∞/c_0 but which contains a complemented isomorphic copy of ℓ_∞/c_0. The Pełczyński decomposition method and the existence of an isomorphism between ℓ_∞/c_0 and $\ell_\infty(\ell_\infty/c_0)$ implies that ℓ_∞/c_0 has the Schroeder-Bernstein property assuming CH. On the other hand, the nonprimariness of ℓ_∞/c_0 would imply that it does not have the Schroeder-Bernstein property as observed in [5]. It could be noted that after the first example of a Banach space without the Schroeder-Bernstein property was given in [7], an example of the form $C(K)$, like all the spaces considered in this paper, was constructed as well (see [11]).

Our results (a) - (c) can also be seen in a different light. It is well-known that assuming CH the space ℓ_∞/c_0 is isometrically universal for all Banach spaces of density not bigger than $\mathfrak{c}$. It has been proved by the authors in [1] that this is not the case in the Cohen model, even in the isomorphic sense. The results (a) - (c) show that $\ell_\infty(c_0(\mathfrak{c}))$ or $\ell_\infty(\ell_\infty/c_0)$ can be added to a recently growing list of Banach spaces that consistently do not embed into ℓ_∞/c_0, see [2], [12], [16] or section 3 of [1]. A new feature of the examples provided in this paper is that they are neither obtained from a well-ordering of the continuum nor a generically constructed object like those in the above mentioned papers.

In Section 2 we present some consequences of the assumption that ℓ_∞/c_0 contains an isomorphic copy of $\ell_\infty(c_0(\lambda))$ for some uncountable cardinal λ and Section 3 contains the key forcing lemma (Lemma 3.1), whose proof is inspired by the proof of A. Dow of Theorem 4.5 of [4] that the boundary of a zero set in $\mathbb{N}^*$ is not a retract of $\mathbb{N}^*$ in the Cohen model.

The undefined notation of the paper is fairly standard. Undefined notions related to set theory and independence proofs can be found in [13] and those related to Banach spaces in [6].

Let us now introduce some particular notation concerning the spaces we consider here. Given $A \subseteq \mathbb{N}$, let us denote by $[A]$ the corresponding equivalence class in $\wp(\mathbb{N})/Fin$, by A^* the corresponding clopen set of $\beta\mathbb{N}$ and by $[A]^*$ the clopen set of $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ corresponding to $[A]$.

Given $x \in \ell_\infty$, let us denote by $[x]$ the corresponding equivalence class in ℓ_∞/c_0. We will use the isometries $\ell_\infty \equiv C(\beta\mathbb{N})$ and $\ell_\infty/c_0 \equiv C(\mathbb{N}^*)$ and identify each bounded sequence with its extension to $\beta\mathbb{N}$ and each class $y = [x]$ of bounded sequences in ℓ_∞/c_0 with the restriction to $\mathbb{N}^*$ of an extension of x to $\beta\mathbb{N}$.

For $m, n \in \mathbb{N}$, $\alpha, \beta \in \lambda$ and $\sigma \in \lambda^A$ for $A \subseteq \mathbb{N}$, let

$$1_{n,\alpha}(m)(\beta) = \begin{cases} 1 & \text{if } (n,\alpha) = (m,\beta) \\ 0 & \text{otherwise,} \end{cases}$$

$$1_\sigma(m)(\beta) = \begin{cases} 1 & \text{if } (m,\beta) \in \sigma \\ 0 & \text{otherwise,} \end{cases}$$

and notice that $1_{n,\alpha}, 1_\sigma \in \ell_\infty(c_0(\lambda))$ and they can be thought of as the characteristic functions of $\{(n,\alpha)\}$ and of the graph of σ respectively.

Some of the problems addressed in this paper were considered in [8] under different set-theoretic assumptions. Unfortunately the forthcomming paper announced there which was to contain the proofs of the statements instead of sketches of the proofs has not appeared as far as now. Also the statements and arguments outlined in [8] on page 303 concerning the Cohen model contradict our results.

2. Facts on isomorphic embeddings of $l_\infty(c_0(\lambda))$ into ℓ_∞/c_0

Lemma 2.1. *Suppose $y \in \ell_\infty/c_0 \setminus \{0\}$ and $A \subseteq \mathbb{N}$ is infinite. Then there is an infinite $B \subseteq A$ and $r \in \mathbb{R}$ such that $|r| \geq \frac{\|y|[A]^*\|}{2}$ and $y|[B]^* \equiv r$.*

Proof. Let $x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty$ be such that $y = [x]$. Since $B' = \{n \in A : |x_n| > \frac{\|y|[A]^*\|}{2}\}$ is infinite and $\{x_n : n \in B'\}$ is bounded, there is an infinite $B \subseteq B'$ such that $(x_n)_{n\in B}$ converges to some $r \in \mathbb{R} \setminus \{0\}$. Notice that $|r| \geq \frac{\|y|[A]^*\|}{2}$ and $y|[B]^* \equiv r$. □

Theorem 2.2. *Assume λ is an uncountable cardinal and $T : \ell_\infty(c_0(\lambda)) \to \ell_\infty/c_0$ is an isomorphic embedding. Then there is $X \in [\lambda]^\lambda$ and for each $(n,\alpha) \in \mathbb{N} \times X$ there is an infinite set $E_{n,\alpha} \subseteq \mathbb{N}$ and $r_{n,\alpha} \in \mathbb{R}$ such that*

$$|r_{n,\alpha}| \geq \frac{\|T(1_{n,\alpha})\|}{2}$$

and if $\sigma \in \lambda^{\mathbb{N}}$ is an injective function such that $Im(\sigma) \subseteq X$, then for all $(n,\alpha) \in \mathbb{N} \times X$,

$$T(1_\sigma)|[E_{n,\alpha}]^* \equiv \begin{cases} r_{n,\alpha} & \textit{if } (n,\alpha) \in \sigma \\ 0 & \textit{if } \alpha \notin Im(\sigma). \end{cases}$$

Proof. For each $n \in \mathbb{N}$ and each $\alpha \in \lambda$, by Lemma 2.1 there is $r_{n,\alpha} \in \mathbb{R}$ and an infinite set $E'_{n,\alpha} \subseteq \mathbb{N}$ such that

$$|r_{n,\alpha}| \geq \frac{\|T(1_{n,\alpha})\|}{2}$$

and $T(1_{n,\alpha})|[E'_{n,\alpha}]^* \equiv r_{n,\alpha}$.

Claim: *For every $(n,\alpha) \in \mathbb{N} \times \lambda$, there is a countable set $X(n,\alpha) \subseteq \lambda$ and an infinite set $E_{n,\alpha} \subseteq^* E'_{n,\alpha}$ such that whenever $\sigma \in \lambda^A$ for some nonempty $A \subseteq \mathbb{N}$ is such that $Im(\sigma) \cap X(n,\alpha) = \emptyset$, then we have*

$$T(1_\sigma)|[E_{n,\alpha}]^* \equiv 0.$$

Proof of the claim: If the claim does not hold, let $(n,\alpha) \in \mathbb{N} \times \lambda$ for which the claim fails. We will carry out certain transfinite inductive construction of length ω_1 which will lead to a contradiction. We will construct for each $\xi < \omega_1$ an infinite set $F_\xi \subseteq \mathbb{N}$, $r_\xi \in \mathbb{R} \setminus \{0\}$ and $\sigma_\xi \in \lambda^{A_\xi}$ for some nonempty $A_\xi \subseteq \mathbb{N}$ such that

(1) $F_\eta \subseteq^* F_\xi \subseteq^* E'_{n,\alpha}$ for all $\xi < \eta < \omega_1$,
(2) $\sigma_\xi \cap \sigma_\eta = \emptyset$ for all $\xi < \eta < \omega_1$,
(3) $T(1_{\sigma_\xi})|[F_\xi]^* \equiv r_\xi$ for all $\xi < \omega_1$.

Given $\xi < \omega_1$, suppose we have already constructed infinite sets $(F_\eta)_{\eta<\xi} \subseteq \wp(\mathbb{N})$, $(r_\eta)_{\eta<\xi} \subseteq \mathbb{R} \setminus \{0\}$ and $\sigma_\eta \in \lambda^{A_\eta}$ for some nonempty $A_\eta \subseteq \mathbb{N}$ and all $\eta < \xi$ as above. Let $F'_\xi \subseteq \mathbb{N}$ be an infinite set such that $F'_\xi \subseteq^* F_\eta$ for every $\eta < \xi$. Since $\Lambda = \bigcup\{Im(\sigma_\eta) : \eta < \xi\}$ is a countable subset of λ, by our hypothesis there is $\sigma_\xi \in \lambda^{A_\xi}$ for some nonempty $A_\xi \subseteq \mathbb{N}$ such that $Im(\sigma_\xi) \cap \Lambda = \emptyset$ and

$$T(1_{\sigma_\xi})|[F'_\xi]^* \not\equiv 0$$

and using Lemma 2.1 find $F_\xi \subseteq F'_\xi$ infinite and $r_\xi \in \mathbb{R} \setminus \{0\}$ such that

$$T(1_{\sigma_\xi})|[F_\xi]^* \equiv r_\xi.$$

This concludes the inductive construction of objects satisfying (1), (2) and (3).

We can now find some $\varepsilon > 0$ for which $R_\varepsilon = \{\xi < \omega_1 : |r_\xi| \geq \varepsilon\}$ is infinite (uncountable, actually) and splitting R_ε into two sets, we may assume without loss of generality that either $r_\xi \geq \varepsilon$ for every $\xi \in R_\varepsilon$ or $-r_\xi \geq \varepsilon$ for every $\xi \in R_\varepsilon$.

Fix $m \in \mathbb{N}$ such that $m \cdot \varepsilon > \|T\|$. Choose $\xi_1 < ... < \xi_m$ in R_ε and notice that $|\sum_{i\leq m} r_{\xi_i}| \geq m \cdot \varepsilon > ||T||$.

Since the σ_{ξ_i}'s are pairwise disjoint, we get that

$$||\sum_{i\leq m} 1_{\sigma_{\xi_i}}|| = 1$$

but

$$||T(\sum_{i\leq m} 1_{\sigma_{\xi_i}})|| \geq ||T(\sum_{i\leq m} 1_{\sigma_{\xi_i}})|[F_{\xi_m}]^*|| \geq |\sum_{i\leq m} r_{\xi_i}| > ||T||,$$

which is a contradiction and completes the proof of the claim.

For each $\alpha \in \lambda$, let $X(\alpha) = \bigcup_{n\in\mathbb{N}} X(n,\alpha)$ and notice that $X(\alpha)$ is a countable subset of λ such that for every $n \in \mathbb{N}$, there is an infinite set $E_{n,\alpha} \subseteq E'_{n,\alpha}$ such that whenever $\sigma \in \lambda^{\mathbb{N}}$ and $Im(\sigma) \cap X(\alpha) = \emptyset$, then we have

$$T(1_\sigma)|[E_{n,\alpha}]^* \equiv 0. \tag{2.1}$$

Now apply the Hajnal free-set lemma (Lemma 19.1, [10]) to obtain $X \subseteq \lambda$ of cardinality λ such that $X(\alpha) \cap X \subseteq \{\alpha\}$ for each $\alpha \in X$. This implies that for distinct $\alpha, \beta \in X$, $\alpha \notin X(\beta)$.

Given $\sigma \in \lambda^{\mathbb{N}}$ which is injective and such that $Im(\sigma) \subseteq X$, notice that for distinct $n, n' \in \mathbb{N}$, $\sigma(n) \notin X(\sigma(n'))$, which guarantees that $Im(\sigma \setminus \{(n, \sigma(n))\}) \cap X(\sigma(n)) = \emptyset$.

Assume $\sigma \in \lambda^{\mathbb{N}}$ is injective and for each $(n, \alpha) \in \omega \times X$ let us consider the two cases. If $(n, \alpha) \in \sigma$, then

$$T(1_\sigma)|[E_{n,\alpha}]^* = T(1_{n,\alpha})|[E_{n,\alpha}]^* + T(1_{\sigma\setminus\{(n,\alpha)\}})|[E_{n,\alpha}]^* \equiv r_{n,\alpha},$$

where the last equality follows from (2.1) and the choice of $E_{n,\alpha}$.

If $\alpha \notin Im(\sigma)$, it follows from (2.1) that

$$T(1_\sigma)|[E_{n,\alpha}]^* \equiv 0.$$

□

Although the above theorem is sufficient for our applications, let us note that it has the following more elegant version.

Corollary 2.3. *Assume λ is an uncountable cardinal and $T : \ell_\infty(c_0(\lambda)) \to \ell_\infty/c_0$ is an isomorphic embedding. Then there is an isomorphic embedding $T' : \ell_\infty(c_0(\lambda)) \to \ell_\infty/c_0$ and for each $(n, \alpha) \in \mathbb{N} \times \lambda$ there is an infinite set $E_{n,\alpha} \subseteq \mathbb{N}$ and $r_{n,\alpha} \in \mathbb{R}$ such that*

$$|r_{n,\alpha}| \geq \frac{\|T'(1_{n,\alpha})\|}{2}$$

and for all $(n, \alpha) \in \mathbb{N} \times \lambda$, if $\sigma \in \lambda^{\mathbb{N}}$, then

$$T'(1_\sigma)|[E_{n,\alpha}]^* \equiv \begin{cases} r_{n,\alpha} & \textit{if } (n,\alpha) \in \sigma \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $X \subseteq \lambda$ of cardinality λ and an infinite set $E_{n,\alpha} \subseteq \mathbb{N}$ and $r_{n,\alpha} \in \mathbb{R}$ for each $(n, \alpha) \in \mathbb{N} \times X$ be as in Theorem 2.2.

Let $(X_n)_{n\in\mathbb{N}}$ be a partition of X into countably many sets of cardinality λ and enumerate each X_n as $X_n = \{\gamma^n_\beta : \beta < \lambda\}$.

Define $S : \ell_\infty(c_0(\lambda)) \to \ell_\infty(c_0(\lambda))$ by

$$S(f)(n)(\beta) = \begin{cases} f(n)(\alpha) & \text{if } \beta = \gamma^n_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Notice that S has the following properties:

- S is an isometric embedding.
- If $\sigma \in \lambda^{\mathbb{N}}$, then $S(1_\sigma) = 1_{s(\sigma)}$ where $s(\sigma) = \{(n, \gamma^n_\alpha) : (n, \alpha) \in \sigma\}$, so that $s(\sigma) \in \lambda^{\mathbb{N}}$ is injective, $Im(s(\sigma)) \subseteq X$ and $\gamma_{n,\alpha} \in Im(s(\sigma))$ if and only if $(n, \alpha) \in \sigma$.

Let $T' = T \circ S$, $E'_{n,\alpha} = E_{n,\gamma^n_\alpha}$ and $r'_{n,\alpha} = r_{n,\gamma^n_\alpha}$. Given $\alpha < \lambda$ and $n \in \mathbb{N}$,

$$|r'_{n,\alpha}| = |r_{n,\gamma^n_\alpha}| \geq \frac{\|T(1_{n,\gamma^n_\alpha})\|}{2} = \frac{\|T(S(1_{n,\alpha}))\|}{2} = \frac{\|T'(1_{n,\alpha})\|}{2}.$$

Also, given any $\sigma \in \lambda^{\mathbb{N}}$ and $(n, \alpha) \in \mathbb{N} \times \lambda$, we have:

- if $(n, \alpha) \in \sigma$, then
$$T'(1_\sigma)|[E'_{n,\alpha}]^* = T(S(1_\sigma))|[E_{n,\gamma^n_\alpha}]^* = r_{n,\gamma^n_\alpha} = r'_{n,\alpha},$$
- if $(n, \alpha) \notin \sigma$, then $\gamma^n_\alpha \notin Im(s(\sigma))$ so that
$$T'(1_\sigma)|[E'_{n,\alpha}]^* = T(1_{s(\sigma)})|[E_{n,\gamma^n_\alpha}]^* = 0.$$

This concludes the proof that T' is the isomorphic embedding with the required properties. □

3. Forcing argument

The next lemma still holds if we replace ω_2 by any regular cardinal λ with $\omega_2 \leq \lambda \leq \kappa$. To simplify the notation we state it in this weaker form, which is sufficient for our purposes.

Lemma 3.1. *Let V be a model of CH, $\kappa \geq \omega_2$ and $\mathbb{P} = Fn_{<\omega}(\kappa, 2)$. In $V^{\mathbb{P}}$, if $(E_{n,\alpha} : (n,\alpha) \in \mathbb{N} \times \omega_2)$ are infinite subsets of $\mathbb{N}$ and for each $\sigma \in \omega_2^{\mathbb{N}}$, B_σ is a subset of $\mathbb{N}$ such that*

$$\forall (n,\alpha) \in \sigma \quad E_{n,\alpha} \subseteq^* B_\sigma,$$

then there is a pairwise disjoint subset $\Sigma \subset \omega_2^{\mathbb{N}}$ of cardinality ω_2 such that $\{B_\sigma : \sigma \in \Sigma\}$ has the finite intersection property, that is, for every $\sigma_1, \dots, \sigma_m \in \Sigma$, $B_{\sigma_1} \cap \dots \cap B_{\sigma_m}$ is infinite.

Proof. In V, for $(n,\alpha) \in \mathbb{N} \times \omega_2$ let $\dot{E}_{n,\alpha}$ be a nice-name for an infinite subset of $\mathbb{N}$.

For each $n \in \mathbb{N}$ and $\alpha \in \omega_2$, let $S_{n,\alpha} = supp(\dot{E}_{n,\alpha})$, which are countable subsets of κ since $\mathbb{P}$ is ccc.

By CH and the Δ-system lemma, we may find pairwise disjoint $(A_n)_{n\in\omega} \subseteq [\omega_2]^{\omega_2}$ such that

- for each $n \in \mathbb{N}$, $(S_{n,\alpha})_{\alpha \in A_n}$ is a Δ-system with root Δ_n.

Let $\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$ and since this is a countable set, by a further thinning out of each A_n, we may assume that

- for every $\alpha \in A_n$, $\Delta \cap (S_{n,\alpha} \setminus \Delta_n) = \emptyset$, i.e. $\Delta \cap S_{n,\alpha} = \Delta_n$.

We may also assume that for each $\alpha < \beta$ in A_n there is a bijection $\pi_{n,\alpha,\beta} : S_{n,\alpha} \to S_{n,\beta}$ such that

- $\pi_{n,\alpha,\beta}|_{\Delta_n} = id$,
- $\pi_{n,\alpha,\beta}(\dot{E}_{n,\alpha}) = \dot{E}_{n,\beta}$ (here $\pi_{n,\alpha,\beta}$ denotes the automorphism of $\mathbb{P}$ obtained by lifting $\pi_{n,\alpha,\beta}$).

Inductively choose, for $\xi < \omega_2$, functions $\sigma_\xi \in \omega_2^{\mathbb{N}}$ such that:

- $\sigma_\xi(n) \in A_n$ for each $n \in \mathbb{N}$,
- for all $\xi < \eta < \omega_2$ and all distinct $n, m \in \mathbb{N}$,
$$(S_{n,\sigma_\xi(n)} \setminus \Delta_n) \cap (S_{m,\sigma_\eta(m)} \setminus \Delta_m) = \emptyset,$$
- for all $\xi < \eta < \omega_2$, $\sup_{n\in\omega} \sigma_\xi(n) < \sigma_\eta(0)$, so that $\sigma_\xi \cap \sigma_\eta = \emptyset$.

For each $\xi < \omega_2$, let $\dot{B}_\xi$ be a name for a subset of $\mathbb{N}$ as in the hypothesis of the lemma, that is, such that

$$\mathbb{P} \Vdash \forall (n,\alpha) \in \check{\sigma}_\xi \quad \dot{E}_{n,\alpha} \subseteq^* \dot{B}_{\sigma_\xi}$$

and let $\dot{h}_\xi$ be a nice-name such that

$$\mathbb{P} \Vdash \dot{h}_\xi : \mathbb{N} \to \mathbb{N} \text{ is such that } \forall (n,\alpha) \in \check{\sigma}_\xi \quad \dot{E}_{n,\alpha} \setminus \dot{h}_\xi(n) \subseteq \dot{B}_{\sigma_\xi}.$$

Let $R_\xi = supp(\dot{h}_\xi)$ and $S_\xi = \bigcup_{n\in\mathbb{N}} (S_{n,\sigma_\xi(n)} \setminus \Delta_n)$ and notice that the S_ξ's are pairwise disjoint countable subsets of κ. By the Δ-system lemma and further thinning out there is $A \subseteq \omega_2$ of cardinality ω_2 such that

- $(R_\xi)_{\xi \in A}$ is a Δ-system with root R,
- for all $\xi \in A$, $\Delta \cap (R_\xi \setminus R) = \emptyset$,
- for all distinct $\xi, \eta \in A$, $S_\xi \cap R_\eta = \emptyset$,

where the last property can be achieved by Hajnal's free set lemma (Lemma 19.1, [10]).

Fix $m \in \mathbb{N}$ and $\xi_1 < ... < \xi_m$ from A and let us prove that

$$\mathbb{P} \Vdash \dot{B}_{\sigma_{\xi_1}} \cap \cdots \cap \dot{B}_{\sigma_{\xi_m}} \text{ is infinite,}$$

which would give that $\{B_{\sigma_\xi} : \xi \in A\}$ has the finite intersection property. Otherwise, there are $p \in \mathbb{P}$ and $l \in \mathbb{N}$ such that

$$p \Vdash \dot{B}_{\sigma_{\xi_1}} \cap \cdots \cap \dot{B}_{\sigma_{\xi_m}} \subseteq \check{l}.$$

Given $n \in \mathbb{N}$ we say that $q \in \mathbb{P}$ is n-symmetric if

$$\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}(q|_{S_{n,\sigma_{\xi_i}(n)}}) = q|_{S_{n,\sigma_{\xi_j}(n)}} \text{ for all } 1 \le i < j \le m.$$

Fix $n \in \mathbb{N}$ such that $dom(p) \cap S_{n,\sigma_{\xi_i}(n)} = \emptyset$ for all $1 \le i \le m$ and notice that p is n-symmetric. Let us find $q \le p$ which is n-symmetric and $k_1, \ldots, k_m \in \mathbb{N}$ such that $q \Vdash \dot{h}_{\xi_i}(\check{n}) = \check{k}_i$. To do so, we will construct conditions $p_m \le p_{m-1} \le \cdots \le p_1 \le p$ and $k_1, \ldots, k_m \in \mathbb{N}$ such that each p_i is n-symmetric and $p_i \Vdash \dot{h}_{\xi_i}(\check{n}) = \check{k}_i$. Let $p_0 = p$.

Given $1 \le i \le m$, let $q_i \le p_{i-1}$ and $k_i \in \mathbb{N}$ be such that $q_i \Vdash \dot{h}_{\xi_i}(\check{n}) = \check{k}_i$ and $dom(q_i) \setminus dom(p_{i-1}) \subseteq R_{\xi_i}$. Let

$$p_i = q_i \cup \bigcup_{1 \le j \le m} \pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}(q_i|_{S_{n,\sigma_{\xi_i}(n)}}).$$

Notice that $p_i \in \mathbb{P}$, p_i is n-symmetric and, since $p_i \le q_i$, $p_i \Vdash \dot{h}_{\xi_i}(\check{n}) = \check{k}_i$, as we wanted.

Now p_m is an n-symmetric condition such that

$$p_m \Vdash \forall 1 \le i \le m \quad \dot{h}_{\xi_i}(\check{n}) = \check{k}_i.$$

Since $\mathbb{P}$ forces that $\dot{E}_{n,\sigma_{\xi_1}(n)}$ is infinite, let $r_1 \le p_m$ and $n_0 > \max\{k_1, \ldots, k_m, l\}$ be such that $r_1 \Vdash n_0 \in \dot{E}_{n,\sigma_{\xi_1}(n)}$ and $dom(r_1) \setminus dom(p_m) \subseteq S_{n,\sigma_{\xi_1}(n)}$. Let

$$r = r_1 \cup \bigcup_{2 \le j \le m} \pi_{n,\sigma_{\xi_1}(n),\sigma_{\xi_j}(n)}(r_1)$$

and notice that $r \in \mathbb{P}$, $r \le p$ and

$$r \Vdash \check{n}_0 \in (\dot{E}_{n,\xi_1} \setminus \dot{h}_{\xi_1}(n)) \cap \cdots \cap (\dot{E}_{n,\xi_m} \setminus \dot{h}_{\xi_m}(n)),$$

so that

$$r \Vdash \check{n}_0 \in \dot{B}_{\sigma_{\xi_1}} \cap \cdots \cap \dot{B}_{\sigma_{\xi_m}},$$

which contradicts our assumption since $n_0 > l$. This concludes the proof. □

Theorem 3.2. *Let V be a model of CH, $\kappa \ge \omega_2$ and $\mathbb{P} = Fn_{<\omega}(\kappa, 2)$. In $V^{\mathbb{P}}$ there is no isomorphic embedding $T : \ell_\infty(c_0(\omega_2)) \to \ell_\infty/c_0$.*

Proof. We work in $V^{\mathbb{P}}$. Let $\varepsilon > 0$ be such that $\frac{1}{\|T^{-1}\|} > \varepsilon$.

By Corollary 2.3 we may assume that for each $(n, \alpha) \in \mathbb{N} \times \omega_2$ there is an infinite set $E_{n,\alpha} \subseteq \mathbb{N}$ and $r_{n,\alpha} \in \mathbb{R}$ such that

$$|r_{n,\alpha}| \ge \frac{\|T(1_{n,\alpha})\|}{2} \ge \frac{1}{2 \cdot \|T^{-1}\|} > \frac{\varepsilon}{2} \tag{3.1}$$

and for all $n \in \mathbb{N}$ and all $\alpha \in \omega_2$, if $\sigma \in \omega_2^{\mathbb{N}}$, then

$$T(1_\sigma)|[E_{n,\alpha}]^* \equiv \begin{cases} r_{n,\alpha}, \text{ if } (n,\alpha) \in \sigma \\ 0, \text{ otherwise.} \end{cases} \tag{3.2}$$

For each $\sigma \in \omega_2^{\mathbb{N}}$, fix any representative $x_\sigma \in C(\beta\mathbb{N})$ of $T(1_\sigma)$ and let

$$B_\sigma = \{k \in \mathbb{N} : |x_\sigma(k)| > \frac{\varepsilon}{4}\}.$$

Then, we get that B_σ's are as in the hypothesis of Lemma 3.1.

Given $m \in \mathbb{N}$ such that $\|T\| < m \cdot \frac{\varepsilon}{4}$, by Lemma 3.1 there are pairwise disjoint $\sigma_1, \ldots, \sigma_{2m} \in \omega_2^{\mathbb{N}}$ such that

$$B = B_{\sigma_1} \cap \ldots \cap B_{\sigma_{2m}} \text{ is infinite.}$$

Given $u \in [B]^*$, let $1 \leq j_1 < \cdots < j_m \leq 2m$ be such that $T(1_{\sigma_{j_i}})(u)$ are either all positive or all negative. Then

$$|T(\sum_{i=1}^{m} 1_{\sigma_{j_i}})(u)| \geq m \cdot \frac{\varepsilon}{4} > \|T\|,$$

which contradicts the fact that

$$\|\sum_{i=1}^{m} 1_{\sigma_{j_i}}\| = 1$$

and concludes the proof. □

Note that apparently we did not use in the above proof the entire strength of Corollary 2.3, namely we do not use the fact that $T(1_\sigma)|E_{n,\alpha} \equiv 0$ when $(n,\alpha) \notin \sigma$. However this is used within the proof of Corollary 2.3 to conclude that $T(1_\sigma)|E_{n,\alpha} \equiv r_{n,\alpha}$ when $(n,\alpha) \in \sigma$.

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Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, 05314-970, São Paulo, Brazil
E-mail address: `christina.brech@gmail.com`

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland
E-mail address: `piotr.koszmider@impan.pl`